Significance

These wires produced magnetic fields of equal magnitude but opposite directions at each other's locations. Whether the fields are identical or not, the forces that the wires exert on each other are always equal in magnitude and opposite in direction (Newton's third law).

12.4 Check Your Understanding Two wires, both carrying current out of the page, have a current of magnitude 2.0 mA and 3.0 mA, respectively. The first wire is located at (0.0 cm, 5.0 cm) while the other wire is located at (12.0 cm, 0.0 cm). What is the magnitude of the magnetic force per unit length of the first wire on the second and the second wire on the first?

12.4 | Magnetic Field of a Current Loop

Learning Objectives

By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a current in a loop of wire at a point along a line perpendicular to thep lane of the loop.
- Determine the magnetic field of an arc of current.

The circular loop of **Figure 12.11** has a radius R, carries a current I, and lies in the xz-plane. What is the magnetic field due to the current at an arbitrary point P along the axis of the loop?

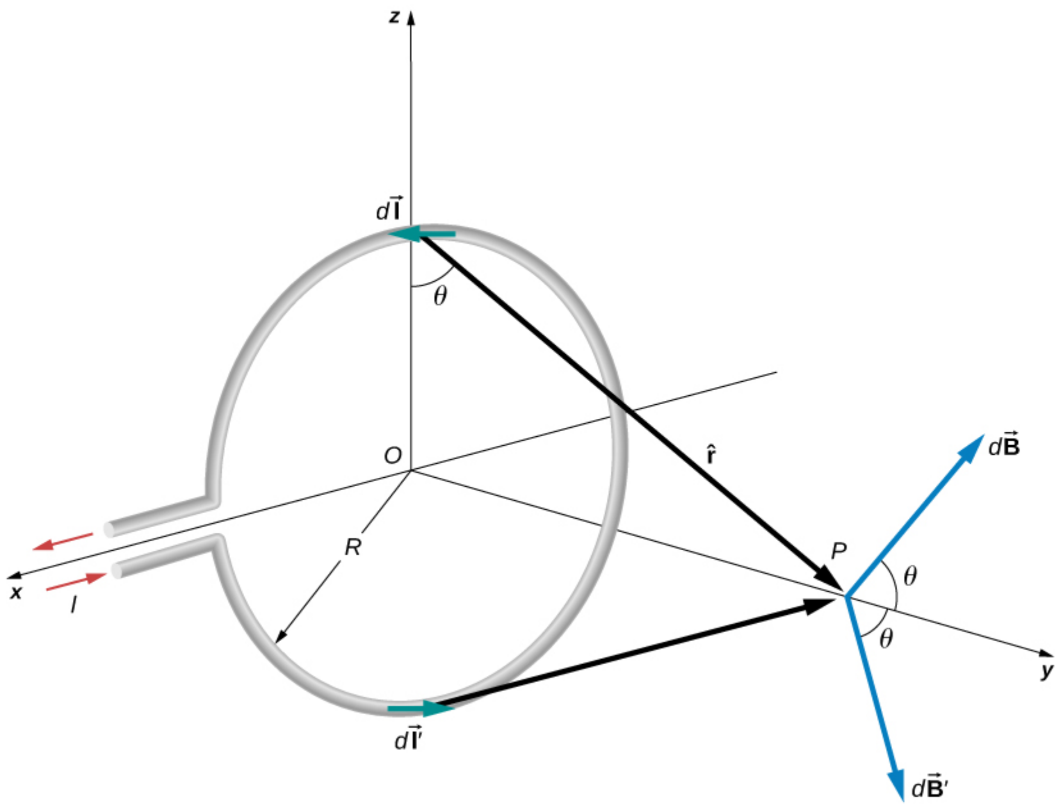

Figure 12.11 Determining the magnetic field at point P along the axis of a current-carrying loop of wire.

We can use the Biot-Savart law to find the magnetic field due to a current. We first consider arbitrary segments on opposite sides of the loop to qualitatively show by the vector results that the net magnetic field direction is along the central axis from the loop. From there, we can use the Biot-Savart law to derive the expression for magnetic field.

Let P be a distance y from the center of the loop. From the right-hand rule, the magnetic field $d\vec{\mathbf{B}}$ at P, produced by the current element $I\,d\vec{\mathbf{l}}$, is directed at an angle θ above the y-axis as shown. Since $d\vec{\mathbf{l}}$ is parallel along the x-axis and $\hat{\mathbf{r}}$ is in the yz-plane, the two vectors are perpendicular, so we have

$$dB = \frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2} = \frac{\mu_0}{4\pi}\frac{I\,dl}{y^2+R^2} \tag{12.13}$$

where we have used $r^2 = y^2 + R^2$.

Now consider the magnetic field $d\vec{\mathbf{B}}\,'$ due to the current element $I\,d\vec{\mathbf{l}}\,'$, which is directly opposite $I\,d\vec{\mathbf{l}}$ on the loop. The magnitude of $d\vec{\mathbf{B}}\,'$ is also given by **Equation 12.13**, but it is directed at an angle θ below the y-axis. The components of $d\vec{\mathbf{B}}$ and $d\vec{\mathbf{B}}\,'$ perpendicular to the y-axis therefore cancel, and in calculating the net magnetic field, only the components along the y-axis need to be considered. The components perpendicular to the axis of the loop sum to zero in pairs. Hence at point P:

$$\vec{\mathbf{B}} = \hat{\mathbf{j}}\int_{\text{loop}} dB\cos\theta = \hat{\mathbf{j}}\frac{\mu_0 I}{4\pi}\int_{\text{loop}}\frac{\cos\theta\,dl}{y^2+R^2}. \tag{12.14}$$

For all elements $d\vec{\mathbf{l}}$ on the wire, y, R, and $\cos\theta$ are constant and are related by

$$\cos\theta = \frac{R}{\sqrt{y^2+R^2}}.$$

Now from **Equation 12.14**, the magnetic field at P is

$$\vec{\mathbf{B}} = \hat{\mathbf{j}}\frac{\mu_0 IR}{4\pi(y^2+R^2)^{3/2}}\int_{\text{loop}} dl = \frac{\mu_0 IR^2}{2(y^2+R^2)^{3/2}}\hat{\mathbf{j}} \tag{12.15}$$

where we have used $\int_{\text{loop}} dl = 2\pi R$. As discussed in the previous chapter, the closed current loop is a magnetic dipole of moment $\vec{\mu} = IA\hat{\mathbf{n}}$. For this example, $A = \pi R^2$ and $\hat{\mathbf{n}} = \hat{\mathbf{j}}$, so the magnetic field at P can also be written as

$$\vec{\mathbf{B}} = \frac{\mu_0\mu\,\hat{\mathbf{j}}}{2\pi(y^2+R^2)^{3/2}}. \tag{12.16}$$

By setting $y = 0$ in **Equation 12.16**, we obtain the magnetic field at the center of the loop:

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2R}\hat{\mathbf{j}}. \tag{12.17}$$

This equation becomes $B = \mu_0 nI/(2R)$ for a flat coil of n loops per length. It can also be expressed as

$$\vec{\mathbf{B}} = \frac{\mu_0\vec{\mu}}{2\pi R^3}. \tag{12.18}$$

If we consider $y \gg R$ in **Equation 12.16**, the expression reduces to an expression known as the magnetic field from a dipole:

$$\vec{\mathbf{B}} = \frac{\mu_0 \, \vec{\mu}}{2\pi y^3}. \tag{12.19}$$

The calculation of the magnetic field due to the circular current loop at points off-axis requires rather complex mathematics, so we'll just look at the results. The magnetic field lines are shaped as shown in **Figure 12.12**. Notice that one field line follows the axis of the loop. This is the field line we just found. Also, very close to the wire, the field lines are almost circular, like the lines of a long straight wire.

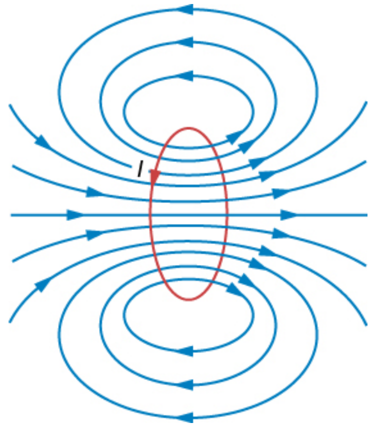

Figure 12.12 Sketch of the magnetic field lines of a circular current loop.

Example 12.5

Magnetic Field between Two Loops

Two loops of wire carry the same current of 10 mA, but flow in opposite directions as seen in **Figure 12.13**. One loop is measured to have a radius of $R = 50\,\text{cm}$ while the other loop has a radius of $2R = 100\,\text{cm}$. The distance from the first loop to the point where the magnetic field is measured is 0.25 m, and the distance from that point to the second loop is 0.75 m. What is the magnitude of the net magnetic field at point *P*?

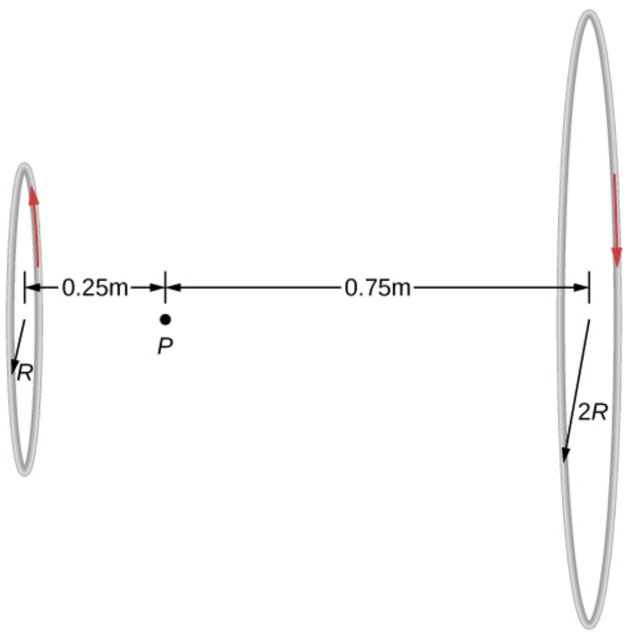

Figure 12.13 Two loops of different radii have the same current but flowing in opposite directions. The magnetic field at point *P* is measured to be zero.

Strategy

The magnetic field at point *P* has been determined in **Equation 12.15**. Since the currents are flowing in opposite directions, the net magnetic field is the difference between the two fields generated by the coils. Using the given quantities in the problem, the net magnetic field is then calculated.

Solution

Solving for the net magnetic field using **Equation 12.15** and the given quantities in the problem yields

$$\begin{aligned} B &= \frac{\mu_0 I R_1{}^2}{2\left(y_1{}^2 + R_1{}^2\right)^{3/2}} - \frac{\mu_0 I R_2{}^2}{2\left(y_2{}^2 + R_2{}^2\right)^{3/2}} \\ B &= \frac{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A})(0.010\,\mathrm{A})(0.5\,\mathrm{m})^2}{2((0.25\,\mathrm{m})^2 + (0.5\,\mathrm{m})^2)^{3/2}} - \frac{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A})(0.010\,\mathrm{A})(1.0\,\mathrm{m})^2}{2((0.75\,\mathrm{m})^2 + (1.0\,\mathrm{m})^2)^{3/2}} \\ B &= 5.77 \times 10^{-9}\,\mathrm{T}\ \text{to the right.} \end{aligned}$$

Significance

Helmholtz coils typically have loops with equal radii with current flowing in the same direction to have a strong uniform field at the midpoint between the loops. A similar application of the magnetic field distribution created by Helmholtz coils is found in a magnetic bottle that can temporarily trap charged particles. See **Magnetic Forces and Fields** for a discussion on this.

12.5 Check Your Understanding Using **Example 12.5**, at what distance would you have to move the first coil to have zero measurable magnetic field at point *P*?

12.5 | Ampère's Law

Learning Objectives

By the end of this section, you will be able to:

- Explain how Ampère's law relates the magnetic field produced by a current to the value of the current
- Calculate the magnetic field from a long straight wire, either thin or thick, by Ampère's law

A fundamental property of a static magnetic field is that, unlike an electrostatic field, it is not conservative. A conservative field is one that does the same amount of work on a particle moving between two different points regardless of the path chosen. Magnetic fields do not have such a property. Instead, there is a relationship between the magnetic field and its source, electric current. It is expressed in terms of the line integral of $\vec{\mathbf{B}}$ and is known as **Ampère's law**. This law can also be derived directly from the Biot-Savart law. We now consider that derivation for the special case of an infinite, straight wire.

Figure 12.14 shows an arbitrary plane perpendicular to an infinite, straight wire whose current *I* is directed out of the page. The magnetic field lines are circles directed counterclockwise and centered on the wire. To begin, let's consider $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ over the closed paths *M* and *N*. Notice that one path (*M*) encloses the wire, whereas the other (*N*) does not. Since the field lines are circular, $\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}}$ is the product of *B* and the projection of *dl* onto the circle passing through $d\vec{\mathbf{l}}$. If the radius of this particular circle is *r*, the projection is $r d\theta$, and

$$\vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = Br\,d\theta.$$